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Chaotic systems with a null conditional Lyapunov exponent under nonlinear driving

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Control of a chaotic system by homogeneous nonlinear driving, when a conditional Lyapunov exponent is zero, may give rise to special and interesting synchronizationlike behaviors in which the response evolves in perfect correlation with the drive. Among them, there are the amplification of the drive attractor and the shift of it to a different region of phase space. In this paper, these synchronizationlike behaviors are discussed, and demonstrated by computer simulation of the Lorentz model [E. N. Lorenz, J. Atmos. Sci. **20** 130 (1963)] and the double scroll [T. Matsumoto, L. O. Chua, and M. Komuro, IEEE Trans. CAS **CAS-32**, 798 (1985)].

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The synchronization of chaotic systems by means of nonlinear driving is a phenomenon, recently discovered [1], that has received considerable attention in the literature of the last few years [2–9], because of both the surprising nature of the phenomenon and its prospective practical applications, in particular in communication technology [3,4] and in neural science [1,5]. When this phenomenon occurs, the phase space trajectory of a conveniently chosen copy of a subsystem of a chaotic system may converge to the trajectory of the original subsystem by driving it with a proper signal from the full chaotic system. This synchronization of the copy subsystem to the original occurs when the conditional Lyapunov exponents of the last are all negative [1,2]. Almost all the literature on this subject deals with this case, while the situation in which there are non-negative exponents has received only marginal attention [6]. However, even when not all the conditional Lyapunov exponents are negative, we might have interesting phenomena because we will be driving systems with chaotic signals. In this paper, I will point out that when all exponents are negative, except for a few that take a zero value, one may find cases in which the copy of the subsystem reproduces its original despite the fact that the distance between both subsystems does not converge to zero. This can be called synchronization, if the term is understood in some generalized sense [7,8]. In particular, I will present two examples, one in which the copy subsystem follows an attractor which is a magnified (or reduced) copy of the original and another in which the copy reproduces the original in a region of phase space where the latter will be unstable.

Following Pecora and Carroll [1,2] one starts with a chaotic system described by the set of variables u that evolves under the equation $\dot{u}=f(u)$. This is divided into two subsystems, described by the sets of variables v and w, such that $u=(v,w)$, and the evolution equations $\dot{v}=g(v,w)$ and $\dot{w}=h(v,w)$ are obeyed. Then duplicate the w subsystem and call w' the new variables, that evolve under the additional equation $\dot{w}'=h(v,w')$. The composed system, of variables (v,w), defines the drive system that controls the evolution of the subsystem, of variables w', known as the response system, through the variables v, called the drive signal. Synchronization occurs when the distance in phase space between the subsystems w and w', $\Delta w(t)=w'(t)-w(t)$, converges to zero as time increases. The time evolution of Δw obeys the variational equation $d(\Delta w)/dt=D_w h(v,w)\Delta w$, where $D_w h(v,w)$ is the Jacobian of the w subsystem. The corresponding Lyapunov exponents are called conditional Lyapunov exponents. If they are all negative there will be stable synchronization of w' to w [1,2].

In this paper, I will address the special case in which one has conditional Lyapunov spectra composed of zero and negative conditional Lyapunov exponents. For simplicity, I will limit the discussion to the three dimensional case; so that $u=(x,y,z)$, $w=(x,y)$, and $v=(z)$. The assumed spectrum of conditional Lyapunov exponents will be of the type $(\lambda_1=0,\ \lambda_2<0)$. A zero conditional Lyapunov exponent would mean that the distance between drive and response remains constant in the average. This may imply that the response does not synchronize at all with the drive; but, it just wanders around it with no correlation between the phase space motion of both systems. However, a zero conditional Lyapunov exponent allows other possibilities which may be of interest from the scientific and technical points of view. The following two basic ones will be studied in this paper. (I) Amplification (or reduction) of the drive: the response describes a trajectory which reproduces the drive, but magnified (or reduced) by a factor A. (II) Phase space shift of the drive: the response reproduces the drive in a region of the phase space which is shifted by a translation vector $\vec{T}=(T_x,T_y)$ from the region where the drive evolves. The values of T_x and T_y, or of A, will depend on the initial conditions of drive and response. Both cases would give rise to a response that, in the average, does not converge nor diverge from the drive; however, the situation differs from the case of no synchronization because here there is an isochronous correspondence between the evolution of the two systems.

A condition for the amplification of the drive to occur is the invariance of the equation $\dot{w}=h(v,w)$ under the coordinate transformation $w^*=Aw$; i.e., $\dot{w}^*=h(v,w^*)$. This can be understood if one has in mind that, for any system, if the initial conditions for drive and response are exactly the same, the response will remain synchronized with the drive. Then, if the above invariance holds, for initial conditions of the response obtained from those of the drive through the above coordinate transformation one will obtain type I synchronization. Likewise, the invariance of $\dot{w}=h(v,w)$ under the

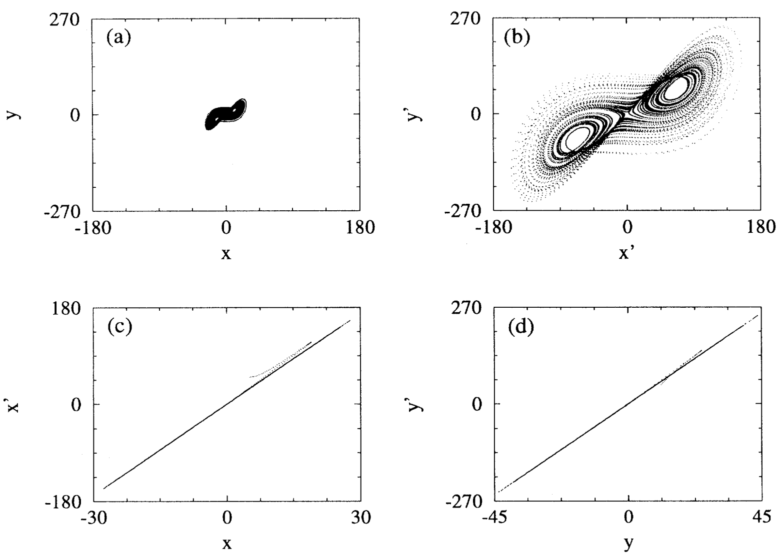

FIG. 1. Syncrhonizaiton behavior of the Lorenz model under z driving: (a) plot of the drive in the x-y plane, (b) plot of the response in the x'-y' plane, (c) parametric plot of the x' signal versus the x signal, and (d) parametric plot of the y' signal versus the y signal.

coordinate translation $w^*=w+T_w$ is a condition for the phase space shift, and other possible symmetries will generate other types of generalized synchronization.

This generalized synchronization differs from Pecora and Carroll's in the stability properties. The synchronization state obtained when all the conditional Lyapunov exponents are negative is stable in the sense that a small perturbation that puts a synchronized response apart from the drive will be rapidly forgotten because the response will converge to the drive. In the case studied here, if a synchronization state of a generalized type is achieved, such perturbation sends the response to another close synchronized state (i.e., it changes the value of A, or of T_x and T_y). This can be so because, with the conditional Lyapunov exponents nonpositive, no exponential divergence of the trajectories of drive and response is to be expected. As a consequence, a small perturbation of the response in an orbit may send it to another orbit, similar and close to the former, where it will stay. In brief, and using the term stability in the sense of Lyapunov [10,11], Pecora and Carroll synchronization is asymptotically stable, while the type of synchronization discussed here is uniformly stable, but not unstable. It is worth noting that uniform stability is still stronger than orbital stability, another important type of stability which is found in planetary and satellite motion [11].

I will demonstrate these ideas by means of a computer simulation study of two models of fluxes. These are the Lorenz model [12] for fluid convective motion

$$\dot{x}=\sigma(y-x),$$
$$\dot{y}=(r-z)x-y, \qquad (1)$$
$$\dot{z}=xy-bz,$$

at the parameter values $r=60$, $\sigma=10$, and $b=8/3$, and the model for an electronic circuit studied by Matsumoto, Chua, and Komuro [13], and known as the double scroll,

$$\dot{x}=\alpha[z-x-G(x)],$$
$$\dot{y}=-\beta z, \qquad (2)$$
$$\dot{z}=x-z+y,$$

where $G(x)$ is a function given by $G(x)=bx+a-b$ if $x\geqslant 1$, $G(x)=ax$ if $|x|<1$, and $G(x)=bx-a+b$ if $x\leqslant -1$, and the parameter values are $\alpha=9$, $\beta=14(2/7)$, $a=-8/7$, and $b=-5/7$. To maintain notation consistent through this paper, when writing Eq. (2), I have interchanged the names of the variables y and z, and the order of the last two equations with respect to the original in [13]. At the parameter values indicated, the two systems are well inside the chaotic regime. I have computed their spectra of conditional Lyapunov exponents under z driving using the method developed by Benettin *et al.* [14], and by Shimada and Nagashima [15]. The results obtained are (0.00, -11.00) for the Lorenz model, and $(0,-1.19)$ for the double scroll. Having in mind the margins of error due to the numerical procedures, these results are in good agreement with the spectrum $(+0.0108,-11.01)$ reported by Pecora and Carroll [1] for the Lorenz model. So, for both models under z driving one exponent is zero and the other is negative. Moreover, each of the sets of equations (1) and (2) exhibits one of the symmetries mentioned above.

In Figs. 1(a) and 1(b) I have displayed trajectories in phase space of drive and response, respectively, for the Lorenz model. The same scale is used in both plots to show how the response follows a trajectory which is an amplification of the drive attractor by a factor of 6 approximately. Parametric plots of the variables of the response versus the variables of the drive are shown in Fig. 1(c) for $x'=x'(x)$ and in Fig. 1(d) for $y'=y'(y)$. The straight lines with slope around 6 are clear evidence that a synchronization state of type I has been achieved. The few points that fall out of this line correspond to an initial transitory. A similar analysis is displayed in Fig. 2 for the double scroll. The drive and response trajectories displayed in Fig. 2(a) show how the response attractor is a copy of the drive displaced a distance

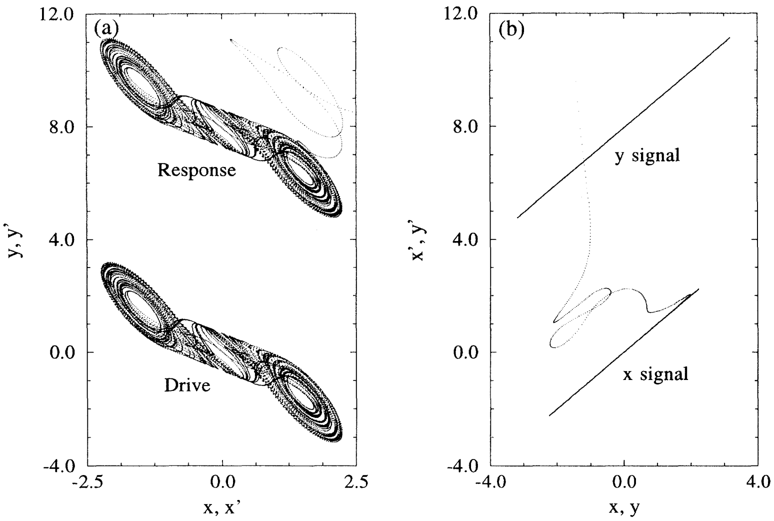

FIG. 2. Synchronization behavior of the double scroll under z driving: (a) plot of the drive and response trajectories, and (b) parametric plots of the response versus the drive signals.

of about 8 units in the y direction. Parametric plots of the response versus the drive variables are displayed in Fig. 2(b). For the x signal a straight line segment centered in the origin and with slope equal to 1 is obtained. For the y signal a segment with slope 1 is also obtained but its center is located around the coordinates (0,8) which proves that generalized synchronization of the type II with $T_x=0$ and $T_y\neq 0$ has been achieved.

To study the effect of the initial conditions, I have computed the time evolutions of drive and response for a fixed initial condition of the drive, chosen as a point in the stable attractor, and a set of initial conditions of the response chosen on a rectangular grid with a linear size one order of magnitude larger than the linear size of the drive attractor. Then, for each of the initial conditions of the drive, I have performed least squares fits to a straight line of the parametric functions $x'=x'(x)$ and $y'=y'(y)$. The slope of this line gives the amplification factor, A, and the ordinates in the origin given the magnitude of the translations, T_x and T_y. For the Lorenz model the function giving the dependence of A on the response initial conditions, (x_0',y_0'), is depicted by a smooth plane that intersects the $A=0$ plane along a straight line that crosses the origin of coordinates, so that the values of A on one side of this line are positive while on the other are negative. The negative values of A appear because of the inversion symmetry of the Lorenz equations around the z axis. This allows two possibilities for the response to reproduce the drive: one with the current state point of both systems evolving in the same quadrant of the x-y plane ($A>0$) and another with the response moving in the quadrant opposed to the one where the drive is currently moving ($A<0$). Moreover, the slope of the plane and the orientation of the line giving its intersection with the $A=0$ plane vary smoothly with the initial condition used for the drive when this is chosen along successive points on the stable attractor. For the double scroll there is also a smooth variation of the value of T_y with the response initial conditions: the function $T_y=T_y(x_0',y_0')$ is depicted by a plane whose intersection with the $T_y=0$ plane is a straight line parallel to the x axis. The slope of this plane does not change with the initial condition of the drive (taken along the stable attractor), while its intersection with the $T_y=0$ plane evolves smoothly. The value of T_x is always zero. Altogether, these results support the assertion that the type of stability in the present case is uniform.

I have studied the amplification and shift of the response when there is imperfect coupling of the drive to the response; i.e., when there is a Gaussian noise added to the drive signal, or there are differences between the parameters of drive and response. In any case, if the imperfection is small enough, the response initially evolves to move in the close neighborhood of the synchronization state it would attain in the case of perfect coupling. However, when there is external noise, because of the uniform nature of stability, one observes a time evolution of A, or of T_w, of a diffusive appearance. The time window needed to observe this evolution decreases from infinity when the level of noise increases from zero. The effect of parameter differences is system dependent because the type of equations that describe the combined system of drive and response are those of five-dimensional homogeneous nonlinear systems, which are used to exhibit complex and varied behaviors. For the Lorenz case, one obtains results very similar to the noisy ones; while, for the double scroll the initial nearly synchronized motion stays and becomes progressively more defective as the parameter mismatch increases. There are important system differences in the sensitivity to external noise, too. For example, for an observation time window that spans over 2000 orbits, the order of magnitude of the variance of the noise at which its presence starts to be noticeable is 10^{-6} for the Lorenz model and 10^{-3} for the double scroll. Anyway, even in the worst case, if the degree of imperfection is small enough, one can observe nearly generalized synchronization states at least inside a time window.

It deserves to be noted that Chua *et al.* [9] made a particular observation of what I have called here type II synchronization in a systematic experimental study on the synchronization of Chua's circuit. The function $G(x)$ for this experimental circuit is somewhat different than the one used here and so are the numerical values of the parameters; however, the spectrum of conditional Lyapunov exponents is still of the type $(0,-)$. This experimental observation, together with the numerical results mentioned in the above paragraph, support the idea that the type of response behavior described here may be robust enough to be sustained by physical systems.

I will finally point out that, besides the results reported, I have observed behavior of this type in the same models at other parameter values, as well as in other mathematical models. I have observed the amplification phenomenon in the Rikitake model for the reversal of Earth's magnetic field [16], the shift of the attractor in two models for electronic oscillators, one by Shinriki, Yamamoto, and Mori [17] and another by Murali and Lakshmanan [4], and a combination of amplification and shift in a model of chemical chaos due to Gaspard and Nicolis [18]. This suggests that, although these type of phenomena are expected for a special type of Lyapunov spectra, they may be more usual than what one might think at first sight.

In conclusion, I have reported theoretical reasoning and numerical evidence of two interesting synchronizationlike phenomena that may happen when a subsystem evolves under nonlinear driving with one of the conditional Lyapunov exponents zero while the other is negative. These are amplification of a chaotic attractor and motion of the response in a region of phase space shifted with respect to the region where the stable attractor evolves. The changes observed in the response state with the variation of initial conditions support the statement that these phenomena may be uniformly stable. This type of generalized synchronization can be observed in mathematical models of a variety of physical systems.

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